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Note on the parameters and the volume element of SU(n)

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Abstract. A convenient parametrization for SU(n) which exhibits the chain of subgroups, SU($n-1$) $\supset$ SU($n-2$) ... $\supset$ SU(1), is proposed and the invariant volume element is calculated.

1. Introduction

The unitary groups have been used extensively to classify multi-electron states of atoms (Judd 1963) and states of nuclei (Jahn and Wieringen 1951, Brink and Nash 1963, Hecht 1965). They have also been used to study the properties of elementary particles (Salam *et al* 1965, Pais 1966) and have, in fact, brought some order in the confusion created by the discovery of too many particles. Considerable attention has, therefore, been given in recent years to the structural properties of these groups and many important results have been obtained. However, in spite of the large number of publications on the subject the only two groups which have so far been studied in detail are SU(2) (Yutsis *et al* 1962) and SU(3). This is primarily due to the fact that the complexity of the computations increases extremely rapidly from SU(2) to SU(3) and, in general, from SU(n) to SU($n+1$).

The purpose of the present note is to derive a few results on SU(n) (Itzykson and Nauenberg 1966, Ciftan and Biedenharn 1969, Mani *et al* 1966) which are valid for all n and which may be of use in calculations involving integrations over the group manifold. First, the group is parametrized in a way which clearly exhibits the chain of subgroups, SU($n-1$) $\supset$ SU($n-2$)... $\supset$ SU(1). The parametrization is, strictly speaking, not new, but is convenient for finding the matrices of finite transformations in an arbitrary irreducible representation of the group. Once the representation matrices are known it becomes possible, in principle, to calculate the Clebsch–Gordan (CG) coefficients by the method of group integration discussed in earlier communications (Majumdar 1969a, 1969b). The invariant volume element required for the integration is derived in § 3.

The groups SU(n) for n = 3, 6, 12 have been used by many authors as possible symmetries of elementary particles. In the study of reactions with two incoming and two outgoing particles belonging to the irreducible representations of one of these groups and in relating the amplitudes in crossed channels (Sharp 1968) the CG coefficients of the group play an important role. Evaluation of these coefficients was the prime motivation behind the present work. No systematic investigation of the CG coefficients of SU(n) has been carried out for values of n beyond 3.

2. Parametrization

To obtain a set of parameters for SU(n) appropriate to the decomposition

$SU(n) \supset SU(n-1) \ldots \supset SU(1)$ we start from Murnaghan's (1962) form of the $U(n)$ matrix:

$$U(n) = \begin{vmatrix} e^{i\delta_1} & 0 \\ 0 & U(n-1) \end{vmatrix} U_{12}(\theta_{n-2}, \sigma_{n-1}) U_{13}(\theta_{n-3}, \sigma_{n-2}) \ldots$$

$$\times U_{1,n-1}(\theta_1, \sigma_2) U_{1n}(\phi_1, \sigma_1). \tag{1}$$

The $U_{rs}(\theta, \sigma)(r < s)$ occurring here are special $SU(n)$ matrices whose nonzero elements are

$$(U_{rs})_{tt} = 1 \qquad \text{for } t \neq r, s; \qquad (U_{rs})_{rr} = (U_{rs})_{ss} = \cos\theta;$$

$$(U_{rs})_{sr} = -(U_{rs})^*_{rs} = \sin\theta\, e^{i\sigma}.$$

We multiply $U_{12}(\theta_{n-2}, \sigma_{n-1})$ on the right by $U_{12}(\frac{1}{2}\pi, 0)$ and apply the transformation $U_{12}(-\frac{1}{2}\pi, 0) U_{1r} U_{12}(\frac{1}{2}\pi, 0)$ to U_{1r} for $r > 2$. (1) is, thus, reduced to

$$U'(n) = \begin{vmatrix} e^{i\delta_1} & 0 \\ 0 & U(n-1) \end{vmatrix} \begin{vmatrix} -\sin\theta_{n-2}\, e^{-i\sigma_{n-1}} & -\cos\theta_{n-2} & 0 & \ldots \\ \cos\theta_{n-2} & -\sin\theta_{n-2}\, e^{i\sigma_{n-1}} & 0 & \ldots \\ 0 & 0 & 1 & \\ \vdots & \vdots & & \ddots \end{vmatrix} \ldots$$

$$\times U_{23}(-\theta_{n-3}, \sigma_{n-2}) \ldots U_{2,n-1}(-\theta_1, \sigma_2) U_{2,n}(-\phi_1, \sigma_1). \tag{2}$$

Next, let P_n be a permutation matrix with 1 in the $(m, m+1)$ positions and $-(-1)^n$ instead of 1 in the $(n, 1)$ position, and let $D_{1r}(\delta)$ be a diagonal matrix with $\exp(i\delta)$ in the $(1, 1)$ position, $\exp(-i\delta)$ in the (r, r) position, and unity in all other positions. Applying the transformation $P_n U'(n) P_n^{-1}$ to $U'(n)$ and making some simple manipulations we then obtain the general $SU(n)$ matrix in the form

$$SU(n) = e^{-i\beta_n Y_n} \begin{vmatrix} SU(n-1) & 0 \\ 0 & 1 \end{vmatrix} e^{-i\mu M} T(n), \tag{3}$$

where

$$T(n) = D_{1r}(\sigma) U_{12}(\psi_2, \alpha_2) \ldots U_{1,n-1}(\psi_{n-1}, \alpha_{n-1}),$$

$$\psi_t = -\theta_{n-t-1}, \qquad \alpha_t = \sigma_{n-t}, \qquad \text{for } t = 2, \ldots, n-2;$$

$$\psi_{n-1} = -\phi_1, \qquad \alpha_{n-1} = \sigma_1,$$

$$\mu = \tfrac{1}{2}\pi + \theta_{n-2}, \qquad \sigma = -(n-\tfrac{1}{2})\pi + \sigma_{n-1}, \tag{4}$$

M is the $U(n)$ generator with unity in $(1, n)$, $(n, 1)$ positions and zeros in all other positions, and Y_n is a diagonal generator with $1/n$ in the first $n-1$ positions and $-(n-1)/n$ in the (n, n) position. From the relations (4) and from Murnaghan's (1962) analysis we see that ψ_{n-1} is a longitude angle and the remaining $2n-2$ parameters explicitly appearing in (3) are latitude angles.

(3) is the desired form (Nelson 1967) of the $SU(n)$ matrix with $\exp(-i\mu M)$ in the middle, two $SU(n-1)$ matrices on the two sides, and the subgroups $SU(r)$ of lower dimensions clearly in evidence. This is, however, not the only possible form of the $SU(n)$ matrix with these properties. One can, clearly, write it in an alternative form with the factors of $T(n)$ decreasing in size from left to right. In fact, by transformations of the type $Q_n SU(n) Q_n^{-1}$ (where Q_n is a permutation matrix with -1 instead of $+1$ at certain places) it is possible to write the $SU(n)$ matrix in a variety of forms (Holland 1969) all

having the desired properties. A particularly interesting form, which has some practical advantages, can be obtained by writing the matrix of the subgroup SU($n-1$) also in the form (2) but arranging its factors in the reverse order. A similarity transformation $U_{23}(-\frac{1}{2}\pi, 0)U_{2r}U_{23}(\frac{1}{2}\pi, 0)$ applied to $U_{2r}(r = 4, \ldots, n)$ on the right hand side then gives the SU(n) matrix a symmetrical form.

Once the SU(n) matrix is cast into one of the above forms it becomes possible to determine the representation matrices of finite transformations. As in the case of SU(3) (Majumdar and Basu 1970) the difficulties of calculation are all contained in $\exp(-i\mu M)$, the only element of SU(n) proper occurring in (3). Since

$$U_{2n}(-\tfrac{1}{2}\pi, 0)\exp(-i\mu M)U_{2n}(\tfrac{1}{2}\pi, 0) = \exp(-i\mu\sigma_1)$$

and $\exp(-i\mu\sigma_1)$ can be represented by ordinary rotation matrices, the problem reduces to the calculation of the matrix elements of $K_n = U_{2n}(-\frac{1}{2}\pi, 0)$. Relations (analogous to (11) and (12) of Majumdar and Basu 1970) between the eigenvalues of the diagonal generators for the initial and the final states of the matrix elements are obtained easily by noting that a traceless diagonal generator after displacement through K_n remains traceless and diagonal. The actual evaluation of the matrix elements of K_n is, however, not so simple and requires a knowledge of the basis states.

3. The volume element

We now proceed to calculate the invariant volume element (VE) of SU(n) in terms of the parameters defined in the previous section. We give small increments to the n^2-1 parameters occurring in the general SU(n) matrix U and carry the infinitesimal matrices (IM) by successive similarity transformations to the position on the extreme right of U. The matrix U, thus, changes to $U(1+\delta U)$. If δU is now expressed as a linear combination of the generators of the group then the coefficient of each generator will represent a row of the Jacobian determinant for the VE. Let us now apply a similarity transformation to δU and express $q\delta Uq^{-1}$ (where q is any element of SU(n)) as a linear combination of the generators. It is easy to see that this will have the effect of multiplying the Jacobian by the matrix $A(q)$ of the adjoint representation of the group. Since the adjoint representation of SU(n) is unimodular the value of the Jacobian remains unaffected by the similarity transformation. This result can be utilized to simplify the calculation of the VE, for, instead of bringing the IM to the right of U we can now bring them to the left of U, or, to a position between any two consecutive factors of U occurring in (3). It is found convenient to choose this position between $\exp(-i\mu M)$ and the factor corresponding to the subgroup SU($n-1$). When all the IM are brought to this position (which will be called 'the position R') and their sum is expressed as a linear combination of the generators, it becomes obvious from an elementary property of determinants that the VE of SU(n) contains that of the subgroup SU($n-1$) as a factor. Thus, one can write, $dV_n = dV_{n-1}\,dG$. To calculate dG we first displace the infinitesimal part of $\exp(-i\beta_n Y_n)$ to the right and bring it to the position R discarding the generators of SU($n-1$) arising in the process. It is easily seen that this IM contributes only an unimportant numerical factor to the VE. Next, we carry the infinitesimal parts of $T(n)$ by successive similarity transformations to the position R and again omit the generators of the SU($n-1$) subgroup, that is, the elements g_{ij} $(i, j \neq n)$ of the resulting matrix $|g|$. As it belongs to the diagonal generators the element g_{nn} can also be omitted. Thus, we are left with the $2n-2$ elements g_{ni}, g_{in} $(i \neq n)$ which alone enter into the calculation of dG.

If the IM assumes the form $\left|\begin{smallmatrix} h & 0 \\ 0 & 0 \end{smallmatrix}\right|$ after reaching the position S immediately before $U_{12}(\psi_2, \alpha_2)$, then after passage through $\exp(-\mathrm{i}\mu M)$ its last row becomes

$$g_{n1} = -\tfrac{1}{2}\mathrm{i} \sin 2\mu (h_{11} + \mathrm{i}s) - \mathrm{i}w, \qquad g_{nr} = -\mathrm{i} \sin \mu \, \mathrm{e}^{2\mathrm{i}\sigma} h_{1r},$$
$$g_{ni} = -\mathrm{i} \sin \mu \, \mathrm{e}^{\mathrm{i}\sigma} h_{1i} \qquad (\mathrm{i} = 2, \ldots, r-1, r+1, \ldots, n-1), \tag{5}$$

where s and w are the infinitesimal increments of σ and μ respectively. The rows of the Jacobian determinant for $\mathrm{d}G$ are formed with the elements g_{ni}, g_{in} $(i \neq n)$ of the matrix $|g|$. It will be seen presently that h_{11} is purely imaginary and $h_{r1} = -h_{1r}^* \, (r = 2, \ldots, n-1)$ are complex quantities with nonvanishing real and imaginary parts. From equation (5) it is, therefore, clear that $\sin 2\mu$ occurs in only one row and $\sin \mu$ in $2(2n-2)$ rows of the determinant. When these μ dependent factors are separated, $\mathrm{d}G$ takes the form

$$\mathrm{d}G = (\sin \mu)^{2n-4} \sin 2\mu \, \mathrm{d}\mu \, \mathrm{d}H \tag{6}$$

where $\mathrm{d}H$ contains only the $2n-4$ variables ψ_m, α_m.

In order to calculate $\mathrm{d}H$ we displace the infinitesimal parts $\mathrm{d}U_{1m}$ of $U_{1m}(\psi_m, \alpha_m)$ to the left until they reach the position S. After the first displacement the IM assumes the form

$$\begin{aligned} \mathrm{d}U_{1m} U_{1m}^{-1} &= -2\mathrm{i} T_3^{(m)} a_m \sin^2 \psi_m + T_+^{(m)} \mathrm{e}^{-\mathrm{i}\alpha_m} (\tfrac{1}{2}\mathrm{i} a_m \sin 2\psi_m - f_m) \\ &\quad + T_-^{(m)} \mathrm{e}^{\mathrm{i}\alpha_m} (\tfrac{1}{2}\mathrm{i} a_m \sin 2\psi_m + f_m) \\ &= 2T_3^{(m)} x_m - T_+^{(m)} y_m^* + T_-^{(m)} y_m, \end{aligned} \tag{7}$$

where the nonzero elements of the matrices $T_3^{(m)}$, $T_\pm^{(m)}$ are

$$(2T_3^{(m)})_{11} = -(2T_3^{(m)})_{mm} = (T_+^{(m)})_{1m} = (T_-^{(m)})_{m1} = 1,$$

and f_m, a_m are the increments of ψ_m, α_m respectively. In carrying the IM further to the left we note that, if any element is created at any stage of the operations outside the first row and the first column, then it remains unaffected by the subsequent operations. Such elements can be discarded as soon as they are created. The determination of the elements in the first row and column is, thus, greatly simplified, and one obtains for the elements in the first column the expressions

$$h_{11} = \sum_{m=2}^{n-1} x_m P_{m-1}^2, \qquad h_{n-1,1} = y_{n-1} P_{n-2},$$
$$h_{r1} = \sum_{m=r+1}^{n-1} x_m B_r P_{m-1}^2 / P_r + y_r P_{r-1} \qquad (r = 2, \ldots, n-2), \tag{8}$$

where

$$P_m = A_2 A_3 \ldots A_m, \qquad B_1 \equiv P_1 \equiv 1, \qquad A_m = \cos \psi_m, \qquad B_m = \mathrm{e}^{\mathrm{i}\alpha_m} \sin \psi_m.$$

The Jacobian for $\mathrm{d}H$ is

$$\frac{\partial(h_{21}, \ldots, h_{n-1,1}, h_{12}, \ldots, h_{1,n-1})}{\partial(a_2, \ldots, a_{n-1}, f_2, \ldots, f_{n-1})}.$$

When h_{r1}, h_{1r} are replaced by the linear combinations, $\mathrm{e}^{-\mathrm{i}\alpha_r} h_{r1} \pm \mathrm{e}^{\mathrm{i}\alpha_r} h_{1r}$, the Jacobian takes a very simple form and is found to have the value

$$(P_2 P_3 \ldots P_{n-2})^2 \sin 2\psi_2 \sin 2\psi_3 \ldots \sin 2\psi_{n-1}.$$

The VE of SU(n), therefore, takes the form

$$\mathrm{d}V_n = \mathrm{d}V_{n-1}|(\sin\mu)^{2n-4}\sin 2\mu\,(\cos\psi_2)^{2n-6}(\cos\psi_3)^{2n-8}\ldots(\cos\psi_{n-2})^2 \times \sin 2\psi_2 \sin 2\psi_3 \ldots \sin 2\psi_{n-1}|\mathrm{d}(\beta_n, \mu, \sigma, \psi_2, \alpha_2, \ldots, \psi_{n-1}, \alpha_{n-1}). \quad (9)$$

For $n = 3$ the formula reduces to $\mathrm{d}V_3 = \mathrm{d}V_2|\sin^2\mu \sin 2\mu \sin 2\psi_2|\,\mathrm{d}(\beta_3, \mu, \sigma, \psi_2, \alpha_2)$. This agrees with the expression obtained previously (Majumdar 1969a, 1969b) for the VE of SU(3) if we take the general element of the SU(2) subgroup to be of the form $\mathrm{e}^{-\mathrm{i}\bar{\alpha}_3 T_3}\,\mathrm{e}^{-\mathrm{i}\bar{\alpha}_2 T_2}\,\mathrm{e}^{-\mathrm{i}\bar{\gamma} T_3}$ and set

$$\beta = \beta_3, \qquad \nu = \mu, \qquad \gamma' = 2\sigma - \alpha_2, \qquad \alpha_2' = -2\psi_2, \qquad \alpha_3' = \alpha_2.$$

The general element of SU(3) then takes the form given by Nelson (1967).

References

Brink D M and Nash G F 1963 *Nucl. Phys.* **40** 608–28
Ciftan M and Biedenharn L C 1969 *J. math. Phys.* **10** 221–32
Hecht K T 1965 *Nucl. Phys.* **62** 1–36
Holland D F 1969 *J. math. Phys.* **10** 1903–5
Itzykson C and Nauenberg M 1966 *Rev. mod. Phys.* **38** 95–120
Jahn H A and Wieringen H van 1951 *Proc. R. Soc.* A **209** 502–24
Judd B R 1963 *Operator Techniques in Atomic Spectroscopy* (New York: McGraw-Hill)
Majumdar S D 1969a *Prog. theor. Phys.* **41** 1382–3
—— 1969b *Prog. theor. Phys.* **42** 1475–6
Majumdar S D and Basu B K 1970 *Ann. Phys., NY* **56** 464–73
Mani H S, Mohan G, Pande L K and Singh V 1966 *Ann. Phys., NY* **36** 285–307
Murnaghan F D 1962 *The Unitary and Rotation Groups* (Washington DC: Spartan Books)
Nelson T J 1967 *J. math. Phys.* **8** 857–63
Pais A 1966 *Rev. mod. Phys.* **38** 215–55
Salam A, Delbourgo R and Strathdee J 1965 *Proc. R. Soc.* A **284** 146–58
Sharp R T 1968 *Phys. Rev.* **168** 1753–5
Yutsis A P, Levinson I B and Vanagas V V 1962 *Mathematical Apparatus of the Theory of Angular Momentum* (Jerusalem: Israel Program for Scientific Translation)